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共形双扭曲积芬斯勒度量的局部对偶平坦性

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摘 要: 设 F_1 和 F_2 分别是光滑流形 M_1 和 M_2 上的芬斯勒度量,共形双扭曲积芬斯勒度量是在乘积流形 $M = M_1 \times M_2$ 上赋予的芬斯勒度量 $F = e^{\sigma}\sqrt{f_2^2F_1^2 + f_1^2F_2^2}$,其中 f_1、f_2 和 σ 分别是 M_1、M_2 和 M 上的正值光滑函数。文章给出了共形双扭曲积芬斯勒度量 F 局部对偶平坦当且仅当 F_1 和 F_2 均局部对偶平坦且 F 为乘积芬斯勒度量。

关键词: 芬斯勒度量;共形双扭曲积;局部对偶平坦;乘积芬斯勒度量

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扭曲积是黎曼几何和芬斯勒几何中构造具有特定曲率性质度量的重要方法,在物理学和广义相对论中有着十分重要的应用[1]。1969年,Bishop等人[1]给出了扭曲积黎曼度量的定义,并用其构造了具有负截面曲率的黎曼度量。1992年,Asanov[2]将扭曲积的概念推广到芬斯勒空间。1998年,Asanov又得到了通过扭曲积描述的相对论模型[3]。2001年,Kozma等人[4]得到了扭曲积芬斯勒度量的嘉当联络表达式,给出了扭曲积芬斯勒流形测地线的微分方程刻画。双扭曲积作为扭曲积的推广,能更好地理解流形的几何性质和拓扑性质。2012年,Peyghan等人[5]给出了双扭曲积芬斯勒流形的黎曼曲率与其分量流形上黎曼曲率之间的关系。

在芬斯勒几何中,Weyl定理表明芬斯勒度量的射影性质和共形性质唯一决定了这个度量的几何性质[6]。1929年,Knebelman[7]首次提出芬斯勒度量共形变换的概念,即Knebelman定理。1976年,Hashiguchi[8]进一步研究了芬斯勒度量的共形变换理论,并证明了两个芬斯勒度量共形相关的充要条件是其对应的度量张量成比例。

2019年,Soleiman等人[9]引入了共形双扭曲积芬斯勒度量的概念,该度量是双扭曲积芬斯勒度量的推广,也是共形芬斯勒度量的推广。他们还得到了共形双扭曲积芬斯勒度量的陈联络、嘉当联络等几何量的表达式,并给出了共形双扭曲积芬斯勒度量是黎曼度量的充要条件。2025年,加依达尔·里扎别克等人[10]给出了共形双扭曲积芬斯勒度量是弱贝瓦尔德度量的充要条件。

2000年,Amari等人[11]引入了局部对偶平坦黎曼度量的概念。2006年,Shen[12]将局部对偶平坦这一概念推广到芬斯勒几何中,并给出了局部对偶平坦芬斯勒度量的微分方程刻画。2010年,Cheng等人[13]给出了刻画局部对偶平坦Randers度量的微分方程,并对局部对偶平坦且具有迷向 S 曲率的Randers度量进行了分类。2011年,Xia[14]给出了 $n(n \geqslant 3)$ 维流形上 (α,β) 度量局部对偶平坦的充要条件。2018年,华义平等人[15]证明了局部对偶平坦且共形平坦的Kropina度量是闵可夫斯基度量。2012年,Peyghan等人[16]证明了双扭曲积芬斯勒度量 F 局部对偶平坦当且仅当 F_1 和 F_2 均局部对偶平坦且扭积函数为常数。文章将探究作为双扭曲积芬斯勒度量推广的共形双扭曲积芬斯勒度量局部对偶平坦的充要条件。

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1 预备知识

设M是n维光滑流形,$(x^1,\cdots,x^n)$为M上的局部坐标。设TM是M的切丛,$(x,y)=(x^1,\cdots,x^n,y^1,\cdots,y^n)$为$TM$上的局部坐标。

定义1[17] 流形M上的芬斯勒度量是一函数$F:TM\to R^+$,满足

(i)F在$\tilde{M}=TM-\{$零截面$\}$上是光滑函数;

(ii)$\forall\lambda\in R^+,F(x,\lambda y)=\lambda F(x,y)$;

(iii)$n\times n$阶黑塞矩阵

$$(G_{\alpha\beta})=\left(\frac{1}{2}\frac{\partial^2F^2}{\partial y^\alpha\partial y^\beta}\right)$$

在$\tilde{M}$上是正定矩阵。

设(M_1,F_1)和(M_2,F_2)分别是m维和n维芬斯勒流形,则$M=M_1\times M_2$是一个$m+n$维芬斯勒流形。

设映射$\pi_1:M\to M_1$和$\pi_2:M\to M_2$表示自然投影,且$\forall x=(x_1,x_2)\in M$,其中$x_1=(x^1,\cdots,x^m)\in M_1$,$x_2=(x^{m+1},\cdots,x^{m+n})\in M_2$,有$\pi_1(x)=x_1$和$\pi_2(x)=x_2$成立。

设$d\pi_1:TM\to TM_1$和$d\pi_2:TM\to TM_2$分别表示由π_1和π_2诱导的切映射。则$\forall y=(y_1,y_2)\in T_xM$,其中$y_1=(y^1,\cdots,y^m)\in T_{x_1}M_1$,$y_2=(y^{m+1},\cdots,y^{m+n})\in T_{x_2}M_2$,有$d\pi_1(x,y)=(x_1,y_1)$和$d\pi_2(x,y)=(x_2,y_2)$成立。记$\tilde{M}_1=TM_1-\{$零截面$\}$,$\tilde{M}_2=TM_2-\{$零截面$\}$,$\tilde{M}=\tilde{M}_1\times\tilde{M}_2\subset TM-\{$零截面$\}$.

文章对指标约定如下,$1\leqslant i,j,k,\cdots,\leqslant m$;$1\leqslant\alpha,\beta,\gamma,\cdots,\leqslant m+n$;$1\leqslant\alpha,\beta,\gamma,\cdots,\leqslant m+n$.

定义2[9] 设(M_1,F_1)和(M_2,F_2)是两个芬斯勒流形,$f_1:M_1\to R^+$和$f_2:M_2\to R^+$是两个光滑函数,$\sigma:M=M_1\times M_2\to R^+$是一个光滑函数。芬斯勒度量$F_1$和$F_2$的共形双扭曲积是在乘积流形$M$上赋予的具有如下形式的芬斯勒度量$F:\tilde{M}\to R^+$

$$F(x,y)=e^{\sigma(\pi_1(x),\pi_2(x))}\sqrt{f_2^2(\pi_2(x))F_1^2(\pi_1(x),d\pi_1(y))+f_1^2(\pi_1(x))F_2^2(\pi_2(x),d\pi_2(y))}\tag{1}$$

其中,$x\in M,y\in\tilde{M}$,f_1和f_2被称为扭曲函数,σ被称为共形因子。一般称F为共形双扭曲积芬斯勒度量。称(M,F)是(M_1,F_1)和(M_2,F_2)关于扭曲函数的共形双扭曲积芬斯勒流形。

当$\sigma=0$,但f_1和f_2均不为常数时,F是双扭曲积芬斯勒度量;当σ不是常数,且$f_1=1$与$f_2=1$有且仅有一个成立时,F是共形扭曲积芬斯勒度量;当σ不是常数,且$f_1=1$与$f_2=1$都成立时,F是共形乘积芬斯勒度量。因此,共形双扭曲积芬斯勒度量是以上三种情形的推广。

设$g_{ij}=\frac{1}{2}\frac{\partial^2F_1^2}{\partial y^i\partial y^j},h_{i'j'}=\frac{1}{2}\frac{\partial^2F_2^2}{\partial y^{i'}\partial y^{j'}}$,则共形双扭曲积芬斯勒度量的基本张量矩阵$(G_{\alpha\beta})$为[9]

$$(G_{\alpha\beta})=\begin{pmatrix}G_{ij}&G_{ij'}\\G_{i'j}&G_{i'j'}\end{pmatrix}=\begin{pmatrix}e^{2\sigma}f_2^2g_{ij}&0\\0&e^{2\sigma}f_1^2h_{i'j'}\end{pmatrix}\tag{2}$$

其逆矩阵$(G_{\alpha\beta})$为[9]

$$(G^{\beta\alpha})=\begin{pmatrix}G^{ji}&G^{ji'}\\G^{j'i}&G^{j'i'}\end{pmatrix}=\begin{pmatrix}e^{-2\sigma}f_2^{-2}g^{ji}&0\\0&e^{-2\sigma}f_1^{-2}h^{j'i'}\end{pmatrix}\tag{3}$$

命题1[17] 设(M,F)是一个芬斯勒流形,根据芬斯勒度量F的正一次齐次性及欧拉定理,有

$$\frac{\partial F^2}{\partial y^\alpha}y^\alpha=2F^2,\frac{\partial^2F^2}{\partial y^\beta\partial x^\alpha}y^\beta=2\frac{\partial F^2}{\partial x^\alpha}\tag{4}$$

命题2[12] 设F是定义在流形M上的芬斯勒度量,则F局部对偶平坦当且仅当对M上任意一点都存在一局部坐标系$(x^1,\cdots,x^n)$使得F满足

$$\frac{\partial^2 F^2}{\partial y^\beta \partial x^\alpha} y^\alpha = 2\frac{\partial F^2}{\partial x^\beta} \tag{5}$$

2 局部对偶平坦性

本节主要研究共形双扭曲积芬斯勒度量的局部对偶平坦性,从而推广文献[16]关于双扭曲积芬斯勒度量局部对偶平坦性的相关结论。

命题3 设F是芬斯勒度量F_1和F_2的共形双扭曲积,则有如下等式成立

$$\frac{\partial F^2}{\partial x^j} = 2e^{2\sigma}\frac{\partial \sigma}{\partial x^j}(f_2^2F_1^2 + f_1^2F_2^2) + e^{2\sigma}\left(f_2^2\frac{\partial F_1^2}{\partial x^j} + \frac{\partial f_1^2}{\partial x^j}F_2^2\right) \tag{6}$$

$$\frac{\partial F^2}{\partial x^{j'}} = 2e^{2\sigma}\frac{\partial \sigma}{\partial x^{j'}}(f_2^2F_1^2 + f_1^2F_2^2) + e^{2\sigma}\left(f_1^2\frac{\partial F_2^2}{\partial x^{j'}} + \frac{\partial f_2^2}{\partial x^{j'}}F_1^2\right) \tag{7}$$

$$\frac{\partial^2 F^2}{\partial y^j \partial x^i} = 2e^{2\sigma}\frac{\partial \sigma}{\partial x^i} f_2^2 \frac{\partial F_1^2}{\partial y^j} + e^{2\sigma} f_2^2 \frac{\partial^2 F_1^2}{\partial y^j \partial x^i} \tag{8}$$

$$\frac{\partial^2 F^2}{\partial y^j \partial x^{i'}} = 2e^{2\sigma}\frac{\partial \sigma}{\partial x^{i'}} f_2^2 \frac{\partial F_1^2}{\partial y^j} + e^{2\sigma} \frac{\partial f_2^2}{\partial x^{i'}} \frac{\partial F_1^2}{\partial y^j} \tag{9}$$

$$\frac{\partial^2 F^2}{\partial y^{j'} \partial x^i} = 2e^{2\sigma}\frac{\partial \sigma}{\partial x^i} f_1^2 \frac{\partial F_2^2}{\partial y^{j'}} + e^{2\sigma} \frac{\partial f_1^2}{\partial x^i} \frac{\partial F_2^2}{\partial y^{j'}} \tag{10}$$

$$\frac{\partial^2 F^2}{\partial y^{j'} \partial x^{i'}} = 2e^{2\sigma}\frac{\partial \sigma}{\partial x^{i'}} f_1^2 \frac{\partial F_2^2}{\partial y^{j'}} + e^{2\sigma} f_1^2 \frac{\partial^2 F_2^2}{\partial y^{j'} \partial x^{i'}} \tag{11}$$

证明 由式(1)易得

$$F^2 = e^{2\sigma}(f_2^2F_1^2 + f_1^2F_2^2)$$

上式两边同时关于x^j求偏导易得式(6)成立。式(6)两边同时关于y^i求偏导可得

$$\frac{\partial^2 F^2}{\partial y^i \partial x^j} = 2e^{2\sigma}\frac{\partial \sigma}{\partial x^j} f_2^2 \frac{\partial F_1^2}{\partial y^i} + e^{2\sigma} f_2^2 \frac{\partial^2 F_1^2}{\partial y^i \partial x^j}$$

对换上式中求和指标i与j,可得式(8)成立。同理可证式(7)及式(9)~式(11)成立。

定理1 设F是芬斯勒度量F_1和F_2的共形双扭曲积,则F局部对偶平坦当且仅当F_1和F_2均局部对偶平坦且F为乘积芬斯勒度量。

证明 根据命题2可知,F局部对偶平坦当且仅当式(5)成立,即下列方程成立

$$\frac{\partial^2 F^2}{\partial y^j \partial x^i} y^i + \frac{\partial^2 F^2}{\partial y^j \partial x^{i'}} y^{i'} = 2\frac{\partial F^2}{\partial x^j} \tag{12}$$

$$\frac{\partial^2 F^2}{\partial y^{j'} \partial x^i} y^i + \frac{\partial^2 F^2}{\partial y^{j'} \partial x^{i'}} y^{i'} = 2\frac{\partial F^2}{\partial x^{j'}} \tag{13}$$

将式(6)~式(11)分别代入式(12)和式(13),并注意到$e^{2\sigma} \neq 0$,上述方程等价于

$$2\frac{\partial \sigma}{\partial x^i} f_2^2 \frac{\partial F_1^2}{\partial y^j} y^i + f_2^2 \frac{\partial^2 F_1^2}{\partial y^j \partial x^i} y^i + \frac{\partial F_1^2}{\partial y^j}\left(2\frac{\partial \sigma}{\partial x^{i'}} f_2^2 + \frac{\partial f_2^2}{\partial x^{i'}}\right) y^{i'} = 2F_2^2\left(2\frac{\partial \sigma}{\partial x^j} f_1^2 + \frac{\partial f_1^2}{\partial x^j}\right) + 4\frac{\partial \sigma}{\partial x^j} f_2^2 F_1^2 + 2f_2^2\frac{\partial F_1^2}{\partial x^j} \tag{14}$$

$$2\frac{\partial \sigma}{\partial x^{i'}} f_1^2 \frac{\partial F_2^2}{\partial y^{j'}} y^{i'} + f_1^2 \frac{\partial^2 F_2^2}{\partial y^{j'} \partial x^{i'}} y^{i'} + \frac{\partial F_2^2}{\partial y^{j'}}\left(2\frac{\partial \sigma}{\partial x^i} f_1^2 + \frac{\partial f_1^2}{\partial x^i}\right) y^i = 2F_1^2\left(2\frac{\partial \sigma}{\partial x^{j'}} f_2^2 + \frac{\partial f_2^2}{\partial x^{j'}}\right) + 4\frac{\partial \sigma}{\partial x^{j'}} f_1^2 F_2^2 + 2f_1^2\frac{\partial F_2^2}{\partial x^{j'}} \tag{15}$$

下面先证明必要性。根据F_1的正一次齐次性及欧拉定理易得

$$\frac{\partial F_1^2}{\partial y^j} y^j = 2F_1{}^2, \quad \frac{\partial^2 F_1^2}{\partial y^j \partial x^i} y^j = 2\frac{\partial F_1^2}{\partial x^i} \tag{16}$$

式(14)两边同时与y^j缩并,并应用式(16),可得

$$4\frac{\partial \sigma}{\partial x^i} f_2^2 F_1^2 y^i + 2f_2^2 \frac{\partial F_1^2}{\partial x^i} y^i + 2F_1^2\left(2\frac{\partial \sigma}{\partial x^{i'}} f_2^2 + \frac{\partial f_2^2}{\partial x^{i'}}\right) y^{i'} = 2F_2^2\left(2\frac{\partial \sigma}{\partial x^j} f_1^2 + \frac{\partial f_1^2}{\partial x^j}\right) y^j + 4\frac{\partial \sigma}{\partial x^j} f_2^2 F_1^2 y^j + 2f_2^2\frac{\partial F_1^2}{\partial x^j} y^j \tag{17}$$

将式(17)右边求和指标j替换为i,则该式可改写为

$$4\frac{\partial\sigma}{\partial x^i}f_2^2F_1^2y^i+2f_2^2\frac{\partial F_1^2}{\partial x^i}y^i+2F_1^2(2\frac{\partial\sigma}{\partial x^{i'}}f_2^2+\frac{\partial f_2^2}{\partial x^{i'}})y^{i'}=2F_2^2(2\frac{\partial\sigma}{\partial x^i}f_1^2+\frac{\partial f_1^2}{\partial x^i})y^i+4\frac{\partial\sigma}{\partial x^i}f_2^2F_1^2y^i+2f_2^2\frac{\partial F_1^2}{\partial x^i}y^i \tag{18}$$

化简整理式(18)易得

$$F_1^2(2\frac{\partial\sigma}{\partial x^{i'}}f_2^2+\frac{\partial f_2^2}{\partial x^{i'}})y^{i'}=F_2^2(2\frac{\partial\sigma}{\partial x^i}f_1^2+\frac{\partial f_1^2}{\partial x^i})y^i \tag{19}$$

同理,根据式(15)也可得式(19)成立。由于$F_1^2\neq0$且$F_2^2\neq0$,由式(19)可得

$$\frac{(2\frac{\partial\sigma}{\partial x^i}f_1^2+\frac{\partial f_1^2}{\partial x^i})y^i}{F_1^2}=\frac{(2\frac{\partial\sigma}{\partial x^{i'}}f_2^2+\frac{\partial f_2^2}{\partial x^{i'}})y^{i'}}{F_2^2} \tag{20}$$

由于上式左边与$(y^{i'})$无关,而右边与(y^i)无关,所以由式(20)可得

$$\frac{(2\frac{\partial\sigma}{\partial x^i}f_1^2+\frac{\partial f_1^2}{\partial x^i})y^i}{F_1^2}=0 \tag{21}$$

$$\frac{(2\frac{\partial\sigma}{\partial x^{i'}}f_2^2+\frac{\partial f_2^2}{\partial x^{i'}})y^{i'}}{F_2^2}=0 \tag{22}$$

上述方程等价于

$$(2\frac{\partial\sigma}{\partial x^i}f_1^2+\frac{\partial f_1^2}{\partial x^i})y^i=0 \tag{23}$$

$$(2\frac{\partial\sigma}{\partial x^{i'}}f_2^2+\frac{\partial f_2^2}{\partial x^{i'}})y^{i'}=0 \tag{24}$$

式(23)两边同时关于y^i求偏导,式(24)两边同时关于$y^{i'}$求偏导,可得下列方程

$$2\frac{\partial\sigma}{\partial x^i}f_1^2+\frac{\partial f_1^2}{\partial x^i}=0 \tag{25}$$

$$2\frac{\partial\sigma}{\partial x^{i'}}f_2^2+\frac{\partial f_2^2}{\partial x^{i'}}=0 \tag{26}$$

下面分三步求解上述方程。

第一步:实际上,式(25)中$i=1,\cdots,m$,因此式(25)是由m个方程组成的方程组,经变形易得

$$f_1^{-1}\frac{\partial f_1}{\partial x^i}=-\frac{\partial\sigma}{\partial x^i} \tag{27}$$

由式(27)可得

$$\ln|f_1|=-\sigma+c_1 \tag{28}$$

其中c_1为常数。由式(28)易得

$$|f_1|=e^{-\sigma+c_1} \tag{29}$$

第二步:式(26)中$i'=m+1,\cdots,m+n$,因此式(26)是由n个方程组成的方程组,用第一步中类似的方法可解得

$$|f_2|=e^{-\sigma+c_2} \tag{30}$$

其中c_2为常数。

第三步:进一步分析求解σ. 由于式(29)左边f_1仅与(x^i)有关,因此式(29)右边$e^{-\sigma+c_1}$仅与(x^i)有关,即σ仅为关于(x^i)的函数。同理,由于式(30)左边f_2仅与$(x^{i'})$有关,因此式(30)右边$e^{-\sigma+c_2}$仅与$(x^{i'})$有关,即σ仅为关于$(x^{i'})$的函数。综上,可得$\sigma=c$(c为常数).

通过以上分析可知,$\sigma=c$,$|f_1|=e^{-\sigma+c_1}$,$|f_2|=e^{-\sigma+c_2}$,再将其代入式(1)可得$F=\sqrt{e^{2c_2}F_1^2+e^{2c_1}F_2^2}$,从本质上来看,此时$F$为$M$上的乘积芬斯勒度量。

又由于σ,f_1,f_2均为常数,式(14)经化简整理可得

$$\frac{\partial^2 F_1^2}{\partial y^j \partial x^i} y^i = 2 \frac{\partial F_1^2}{\partial x^j} \tag{31}$$

这意味着 F_1 局部对偶平坦。同理,由于 σ, f_1, f_2 均为常数与式(15)结合可得

$$\frac{\partial^2 F_2^2}{\partial y^{j'} \partial x^{i'}} y^{i'} = 2 \frac{\partial F_2^2}{\partial x^{j'}} \tag{32}$$

这意味着 F_2 局部对偶平坦。

再证明充分性。由 F_1 局部对偶平坦可知式(31)成立。式(31)两边同时乘以 f_2^2,可得

$$f_2^2 \frac{\partial^2 F_1^2}{\partial y^j \partial x^i} y^i = 2 f_2^2 \frac{\partial F_1^2}{\partial x^j} \tag{33}$$

由 F_2 局部对偶平坦可知式(32)成立。式(32)两边同时乘以 f_1^2,可得

$$f_1^2 \frac{\partial^2 F_2^2}{\partial y^{j'} \partial x^{i'}} y^{i'} = 2 f_1^2 \frac{\partial F_2^2}{\partial x^{j'}} \tag{34}$$

又因为 F 为乘积芬斯勒度量,即 $\sigma = 0$,且 f_1 和 f_2 为常数,再结合式(33)和式(34),易验证式(14)和式(15)成立,即 F 局部对偶平坦。

3 结论

文章主要研究共形双扭曲积芬斯勒度量的局部对偶平坦性。局部对偶平坦的芬斯勒度量在信息几何、相对论中具有重要应用。文章证明了共形双扭曲积芬斯勒度量 F 局部对偶平坦当且仅当 F_1 和 F_2 均局部对偶平坦且 F 为乘积芬斯勒度量,由此推广了双扭曲积芬斯勒度量局部对偶平坦的相关结果。

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Research and Practice of Building a Blended Teaching Mode of Analytical Chemistry based on Micro-teaching Platform

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Abstract: To investigate the practical effect of the blended teaching model based on the micro-teaching platform in the teaching of analytical chemistry, a total of 79 students from the College of Chemistry and Chemical Engineering of Xinjiang Normal University were selected as research subjects for the development and practical comparison study of micro-assisted teaching in the teaching of analytical chemistry. The control group was instructed using the conventional teaching methodology, whereas the experimental group was taught through a blended teaching approach based on micro-assisted teaching. Analyzing the chapter test scores of the students in the two classes, it is found that the average scores of the students in the experimental class improved significantly, indicating that the blended teaching model based on micro-assisted teaching effectively promoted the students' mastery of basic knowledge, as well as the improvement of their inferential and computational abilities. Questionnaires and interviews were used to assess the students in the experimental class, and it was found that the combination of student independent learning and teacher guidance in the blended teaching mode significantly improved the students' initiative and enthusiasm in learning. Therefore, the blended teaching model based on micro-assisted teaching not only helps to cultivate students' core literacy, but also strengthens students' confidence and willingness to learn analytical chemistry by constructing a learning framework centered on independent learning.

Keywords: Analytical chemistry; Micro-teaching platform; Blended teaching model

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Locally Dually Flatness Metrics of Conformally Doubly Warped Product Finsler Metrics

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Abstract: Let F_1 and F_2 be Finsler metrics on smooth manifolds M_1 and M_2 respectively, the conformally doubly warped product Finsler metric is a Finsler metric $F = e^{\sigma}\sqrt{f_2^2 F_1^2 + f_1^2 F_2^2}$ defined on the product manifold $M = M_1 \times M_2$, where f_1, f_2 and σ are positive smooth functions on M_1, M_2 and M respectively. In this paper, it is proved that the conformally doubly warped product Finsler metric F is locally dually flat if and only if both F_1 and F_2 are locally dually flat and F is a product Finsler metric.

Keywords: Finsler metric; Conformally doubly warped product; Locally dually flat; Product Finsler metric